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Storage Capacity of Visual Reference Tags

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Abstract

Visual reference (REF) tags are q-state matrix codes designed for an optimal capture and decoding whilst the tagged objects or persons are moving relative to a standard digital camera. We calculate analytically the storage capacity of different visual reference tags on the honeycomb lattice. REF tags implement q-state Potts models with nearest and next-nearest neighbor exclusion. The analytic calculations are performed within the formalism of statistical mechanics. The storage capacity of several tags are computed in analytic form as a function of q.

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I. INTRODUCTION

Visual reference tags are a new type of colored matrix codes. This paper presents the calculation of their storage capacity. The image processing algorithms used for an optimal capture of the codes will be described elsewhere. The Author believes that this is a prticularly simple example for physics and computing science students to get familiar with some methods of statistical physics found useful for calculations in information theory.

The Potts model is a generalization of the Ising model. Similarly to the "Ising" model, which was suggested to his student Ising by Lenz, the Potts model was defined by C. Domb as a generalization of the two-state (binary) Ising model. The q-state Potts model (in strict sense, as used here) is a special case of the general $Z(q)$ class of models. The 'spin' variables in this model form a Z_q Abelian group. Those who want more detailed information about the multiple and interesting applications of the Potts model ought to consult the excellent review paper by Fred Wu [1].

In statistical physics a given model is defined by its energy functional, or Hamiltonian. First, one defines a q-state vertex variable as $l_i = \{0, 1, 2, \ldots q-1\}$ variable. Alternately, the possible values of l_i are called *colors*. One can visualize this variable as a watch hand, pointing at the one of the q equally spaced dots. Next, the interaction between two variables, l_i and l_j is defined by the two-site Hamiltonian as:

$$h(i,j) = -J\delta(l_i, l_j) \tag{1}$$

where the Kronecker symbol $\delta(l_i, l_j) = 1$ if $l_i = l_j$ and $\delta(l_i, l_j) = 0$ otherwise. Similarly, the complementary Kronecker delta can be defined as $\hat{\delta}(l_i, l_j) = 1 - \delta(l_i, l_j)$ and equals 1 whenever $l_i \neq l_j$ and 0 otherwise. Note that if $J > 0$ the interaction energy is lowered when the two variables are parallel ("ferromagnet"), while this state is avoided when $J < 0$ (Potts "antiferromagnet"). We prefer to call the antiferromagnetic case an exclusion model, because it expresses that the two interacting variables avoid having the same value (or color).

A full Hamiltonian (the energy of the system) requires furthermore a lattice (or a graph G) whose vertices(V) will carry the spin variables and the neighborhoud relations between them, through the edges E of the graph. Let us denote the graph by $G(V, E)$ and its energy

by

$$H(G) = \sum_E h(E) = \sum_E h(l_{E(V_1)}, l_{E(V_2)}) \tag{2}$$

where the two variables in h are the two vertices connected by the graph G edge E.

In statistical mechanics the first quantity of interest is the (canonical) partition function, which is the normalization factor of the Boltzmann distribution:

$$Z(T, G) = \sum_{\{l_i\}} e^{KH(G)} \tag{3}$$

where the summation is over all possible spin configurations and $K = -\frac{J}{k_B T}$ with k_B the Boltzmann constant and T the absolute temperature in °K. In general such sums cannot be evaluated analytically, with the notable exception of "exactly soluble" models. Instead, one uses approximate methods, like low (high) temperature series expansions or mean field approximations in order to evaluate the partition function and other quantities of interest. In this paper we use elementary combinatorics, high temperature expansions, and analytic solutions of transfer matrices.

A rather trivial identity to 'linearize' the two-site Boltzmann factor is:

$$e^{K\delta(l_i,l_j)} = 1 + v\delta(l_i, l_j) \tag{4}$$

where a simple calculation for the two $l_i = l_j$ and $l_i \neq l_j$ cases delivers for $v = e^K - 1$. Therefore, since the Hamiltonian (2) involves only sums of two-site contributions of the type (4), more complex Boltzmann distributions will contain products of such linearized terms. As an example consider now the case

$$\sum_{l_j=0}^{q-1} e^{K(\delta(l_i,l_j)+\delta(l_j,l_k))} = \sum_{l_j=0}^{q-1} (1 + v\delta(l_i, l_j))(1 + v\delta(l_j, l_k)) = q + 2v + v^2\delta(l_i, l_k) \tag{5}$$

For visual reference tags one must take the limit $T \to 0$ for the $J < 0$ case, leading to $v = -1$.

II. THE ONE DIMENSIONAL CHAIN

We are now ready to display different solutions techniques on the simple example of a one dimensional chain. The chain consist of N linearly linked edges, all coupled to an external field in state L. Hence,

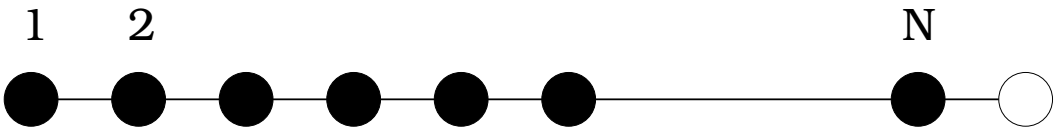

FIG. 1. *A one dimensional chain with free boundary conditions.*

$$H(G) = \sum_{i=1}^{N} [K\delta(l_i, l_{i+1}) + B\delta(l_i, L)] \tag{6}$$

Consider first the recursion method, which can be applied for free boundary conditions, when the last variable (l_{N+1}) in the chain is connected solely to the previous one. Then, one can write (see Fig. 1) as:

$$Z_{N+1}(q) = [q + v + w + vw\delta(l_N, L)]\, Z_N = A\,[1 + b\delta(l_N, L)]\, Z_N = A\tilde{Z}_N \tag{7}$$

where $w = e^B - 1$, $A = q + v + w$, $b = \frac{vw}{A}$ and $\tilde{Z}_N$ differs from Z_N only in that in its product expansion involving the spin l_N, $(1 + w\delta(l_N, L))$ is now multiplied by $\tilde{w} = (w + b + wb)$ instead of w.

This method generates thus a simple recursion on the "external field" linearized factor and a constant factor contributing to the partition function. If $B = 0$, implying $w = 0$:

$$Z_{N+1}(q) = q(q + v)^N = q(q - 1 + e^K)^N \tag{8}$$

If the external field is infinitely strong but antiferromagnetic ($w = -1$) its effect is to forbid one of the states, so that

$$Z_{N+1}(q|K, B = -\infty) = Z_{N+1}(q - 1|K) \tag{9}$$

The second method is called "divide and conquer" in computer science and the "iteration-decoration" (exact) transformation in statistical physics. Assuming that the lattice has $N = 2^{2^k}$ sites one can rewrite the partition function as:

$$Z_{2n}(q) = \sum_{l_1=0}^{q-1} \sum_{l_2=0}^{q-1} \cdots \sum_{l_i=0}^{q-1} \cdots \sum_{l_{2n}=0}^{q-1} \prod_i [1 + v\delta(l_i, l_{i+1})]\, [1 + w\delta(l_i, L)] \tag{10}$$

where we have multiplied an edge weight with the external field contribution, both in linearized form. The main idea is to sum up first every second (even) variable and relate the

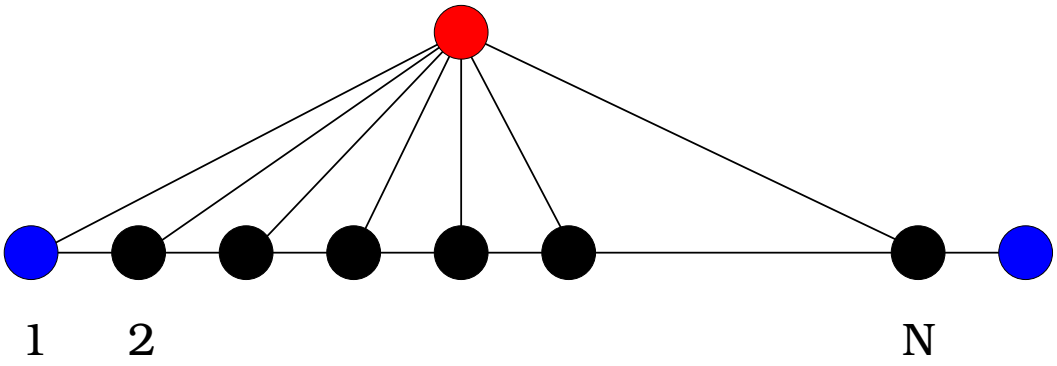

FIG. 2. *A one dimensional chain with perodic boundary conditions in an external field.*

$Z_{2n}(q)$ to $Z_n(q)$, while the interaction form and factors are "renormalized", leading to a finite number of recursion relations (here four). What happens if we wish to sum up l_i? Since all neighbors of l_i are fixed, write down all products in which the variable l_i appears and then sum over l_i:

$$\sum_{l_i} [1 + v\delta(l_{i-1}, l_i)]\,[1 + v\delta(l_i, l_{i+1}]\,[1 + w\delta(l_i, L)] =$$
$$q + 2v + w + v^2\delta(l_{i-1}, l_{i+1}) + vw\,[\delta(l_{i-1}, L) + \delta(l_{i+1}, L)] + v^2 w\delta(l_{i-1}, l_{i+1}, L) \qquad (11)$$

The expression has four coefficients, multiplying different operator structures in the variables l_{i-1}, l_{i+1}, and L. After factorizing out the constant term, one recognize terms corresponding to edge-type interaction, additional external field interactions, and a new operator type, linking both ends of the edge to the external field. This structure is stable under further iterations and leads to three recursions and one dependent constant contribution to the partition function. In computational terms, the recursion algorithm requires $2O(N)$ steps, while the iteration (or 'decimation') method requires only $4o(\log_2 N)$ steps. Can one do even better?

The third method is known in statistical physics as the transfer matrix method and in computer science as the dynamic programming, alias Viterbi trellis method. The main idea is to introduce a direct product of the orthonormal basis for l_i, $\mathbb{1} = \sum_{n=0}^{q-1} |n\rangle\langle n|$, $\langle n|m\rangle = \delta(n, m)$. This method works easiest if the chain is closed by linking the end vertex to the first one, so that we have in total N edges (bonds) and N spins. Taking for instance the kernel of the matrix i to be the product shown in Eq. 10,

$$T_{i,i+1} = \langle l_{i+1}, l_i|\hat{T}|l_i, l_{i+1}\rangle = [1 + z\delta(l_i, L)]\,[1 + v\delta(l_i, l_{i+1})]\,[1 + z\delta(l_{i+1}, L)] \qquad (12)$$

where the choice $z = e^{\frac{B}{2}} - 1$ symmetrizes the transfer matrix and

$$Z_N = Tr(\hat{T}^N) = \sum_{\alpha=0}^{q-1} \lambda_\alpha^N \tag{13}$$

λ_α are the q real eigenvalues of the matrix T.

To illustrate the power of the transfer matrix, consider the chain in Fig. 2 with periodic (cyclic) boundary conditions without an external field at the $T = 0$ limit. Then the transfer matrix looks like

$$T = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ . & . & . & . & . & . \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix} \tag{14}$$

where the zeros on the diagonal express the exclusion condition $l_i \neq l_{i+1}$. This $q \times q$ matrix can be easily diagonalized analytically with a discrete Fourier transformation. The spectrum consists of one single large eigenvalue $t_0 = q-1$ and a set of degenerate eigenvalues $t_\alpha = -1,\ \alpha = 1, \dots, (q-1)$. The partition function is thus

$$Z_N(pbc) = Tr(\hat{T}^N) = t_0 + \sum_{\alpha=1}^{q-1} t_\alpha^N = [q-1]^N + (q-1)(-1)^N \tag{15}$$

to be compared with the free boundary condition solution $Z_N(fbc) = q(q-1)^{N-1}$ from Eq. (8) at $K = -\infty$.

The transfer matrix method requires then only the computation of the q eigenvalues. In general, this calculation does not depend any longer on N and requires numerically $O(q^3)$ steps. If N is quite large, then the main contribution comes from the largest eigenvalue λ_0, which can be calculated by strongly converging iterative methods. There are also many cases when the transfer matrix can be solved analytically, as in the example above. Except for this last case, the dependence on the number of states, q, enters now as the dimension of the matrix and is not any longer an arbitrary parameter of the partition function.

III. THE CORE VISUAL REFERENCE TAG

The core reference tag consists of seven hexagons and is shown in Fig. 3. It consists of six hexagons surrounding a central site. The rules for this central part (core) are quite simple:

1. The tag must contain at least 6 (six) different colors, placed at equal angular distance according to their hue. One standard choice is to use the primary and secondary colors.

2. The seventh color is either black or white.

The rules can be fulfilled only if the total number of states is at least 7 and include **six** real colors. Computing the storage capacity of the core is easy. Let us assume first that the black or white color is on the outer layer. The number of possible configurations according to the rules (1-2) is then $\Omega = 2 \times 6! = 1440$. Since one has only one gray-level hexagon on the outside layer, one can use it as standard starting point and continue the codec always clockwise. Hence, this part is always rotational invariant.

If the black/white site happens to be in the center, one has a similar number of configurations, except the code is not any longer rotational invariant. For such codes the total number of states is 2880. If rotational invariance is required, the number of states of the second half must be divided by 6, since one needs to set one reference color as codec starting point. Thus, for rotational invariant coding the total number of states is $\Omega = 1440 + 240 = 1680$.

If the **number of real colors** is larger than six ($\hat{q} = q - 2 > 6$), then

$$\Omega = 2 \times \left(\frac{\hat{q}!}{(\hat{q}-6)!} + \frac{(\hat{q}-1)!}{(\hat{q}-6)!} \right) \tag{16}$$

When capturing this tag via a digital camera several types or errors might occur. The first (unrecoverable) error is that the tag is not found at all. The second type of error is that because of occlusion or other defect, one site's color is not visible. This is called an erasure error. If one real color site is missing then one can recover it as long as $\hat{q} = 6$. If the black/white site is missing, then the decoder will be presented with a probabilistic estimate of white/black presence. Another type of error can occur if the data colors are too close to each other in hue values. In that case, the decoder is again presented with a probabilistic estimate of 4 possible states: the data state α and the nearest neighbor colors $\alpha \pm 1$, in both alternate positions.

In fact, if the core cannot be correctly decoded the other parts of a tag are probably also erroneous. This will happen only in very noisy environments. In such cases, the coded string (or integer) should be encoded using some effective error correcting code, which acts then in addition to the face-exclusion rules used above.

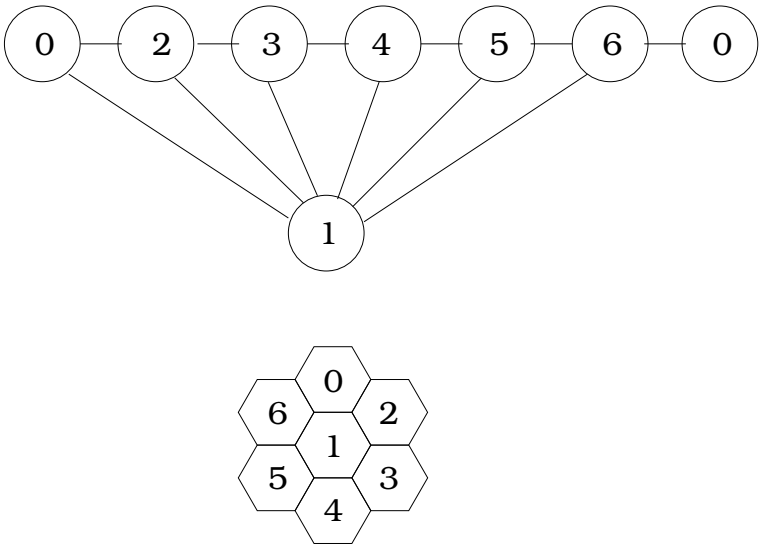

FIG. 3. *The core tag is shown on the lower part, together with one possible enumeration path of the units. The same core is also shown as a one dimensional chain of 6 units with periodic boundary conditions and a common external field on the upper part of the Figure.*

IV. VISUAL REFERENCE TAGS WITH 2 LAYERS

The next type of tag has two layers (a core and an additional layer), as shown in Fig. 4. While the core rule is to exclude any configuration with two identical colors and include all real colors, the second layer fulfills the more relaxed nearest-neighbor rule, requiring that no nearest neighbor pair exhibits the same color. This condition is also known as the map- or face coloring rule in graph theory. Hence, as shown by the upper part of the Figure, one has a linear chain of 12 sites connected to some external exclusion fields corresponding to the actual core coloring and obeying periodic boundary conditions. q denotes the total number of colors used, *including* the white and black colors. The rest of this Section shows how to solve exactly such a problem.

Consider the periodical pattern illustrated by the magenta (8, 10) sites with a yellow site (9) sandwiched inbetween in the upper part of Fig. 4. We use now the iteration procedure discussed in the Introduction to sum up exactly all odd numbered (yellow) sites on the upper row.

Using the notation $v = e^{-K} - 1$ and $a = 8$, $c = 10$, $x = 2$ as shown in Fig. 4, for example,

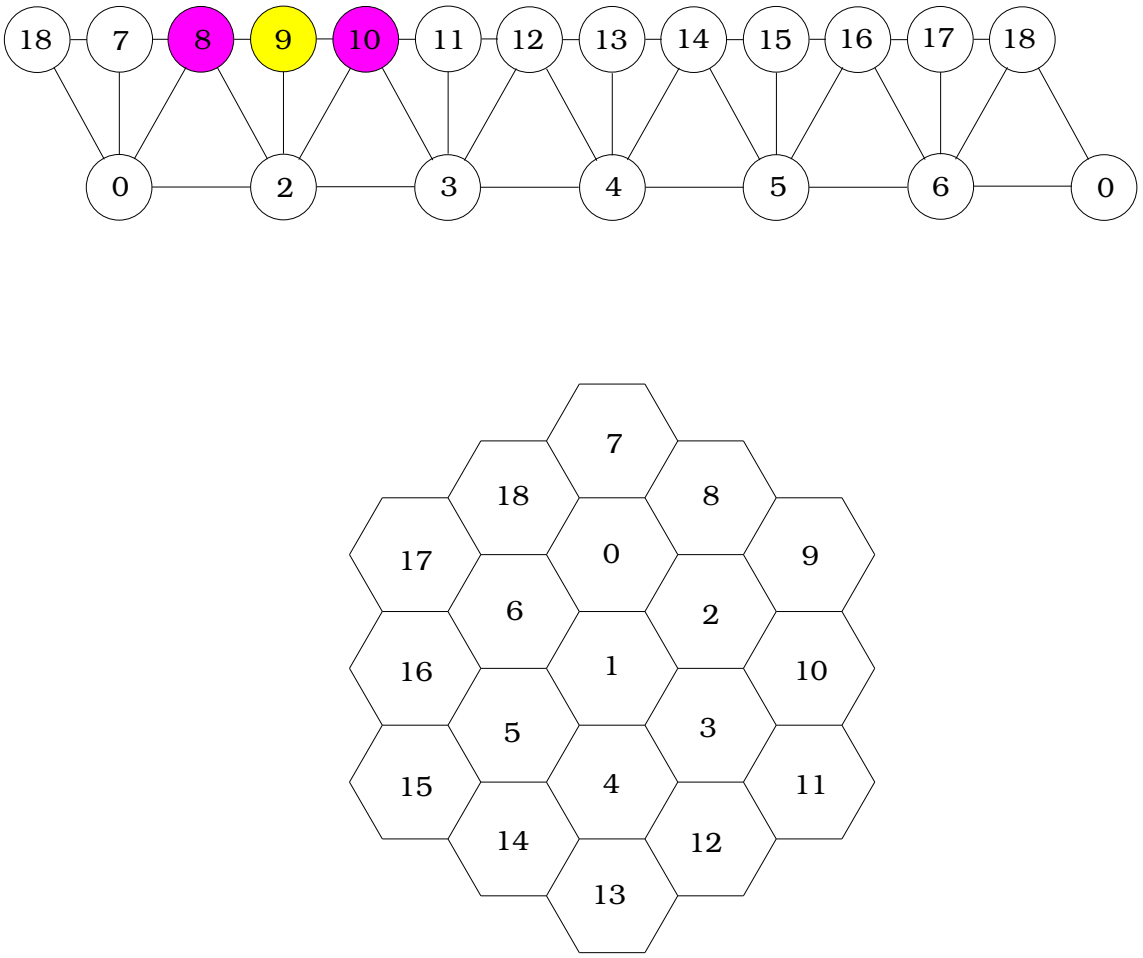

FIG. 4. *The two layers tag is shown on the lower part. The upper part shows a reinterpretation of the tag as a one dimensional chain coupled to six different external fields.*

we obtain for one such summation over $b = 9$

$$k(a,c,x) = v^2(q+v^3+5v^2+8v)\delta(a,c,x)+v^2\delta(a,c)+v(q+v^2+4v)(\delta(a,x)+\delta(c,x))+q+3v \tag{17}$$

which for $T \to 0$ ($v = -1$) reads

$$k(a,c,x) = (q-4)\delta(a,c,x) + \delta(a,c) - (q-3)(\delta(a,x)+\delta(c,x)) + q - 3 \tag{18}$$

The kernel $k(a,c,x)$ contains the factor relating the "external field" x to its neighboring (magenta) sites a and b and represents the effective interaction between two nearest neighbor magenta sites. Let us generalize the kernel $k(a,c,x)$ in Eq. (18) as following

$$K(a,c,x) = A + B\delta(a,c) + C[\delta(a,x)+\delta(c,x)] + D\delta(a,c,x) \tag{19}$$

Next, in order to sum the remaining 6 variables, construct the transfer matrix between the sites a and c, where now x is considered to be a fixed parameter. Without loss of generality, let us assume that the x variable is in the l_0 state. The $q \times q$ transfer matrix between l_a and l_c then reads

$$T_0 = \begin{pmatrix} A+B+2C+D & A+C & A+C & A+C & \dots & A+C \\ A+C & A+B & A & A & \dots & A \\ A+C & A & A+B & A & \dots & A \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A+C & A & A & A & \dots & A+B \end{pmatrix} \tag{20}$$

From symmetry arguments and direct calculations valid in the $T \to 0$ ($v = -1$) exclusion limit, it turns out that both $A+B+2C+D = 0$ and $A+C = 0$, so that in this limit the matrix has the following structure:

$$T_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & A+B & A & A & \dots & A \\ 0 & A & A+B & A & \dots & A \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & A & A & A & \dots & A+B \end{pmatrix} \tag{21}$$

This matrix consists of two blocks: the first nill row and column and the rest submatrix (SM). The matrix is symmetric, so it must have real eigenvalues and an orthogonal system of eigenvectors. From these, the eigenvalues and eigenvectors of SM are easy to compute, since the matrix is a polynom of the elementary cyclic matrix. The remaining eigenvector must be orthogonal to all SM eigenvectors. Denoting $\hat{\omega} = e^{\frac{2\pi i}{(q-1)}}$, the eigenvalues corresponding to the transfer matrix read:

$$\lambda_0 = B + (q-1)A \tag{22}$$

$$\lambda_{1\dots q-2} = B \tag{23}$$

$$\lambda_{q-1} = 0 \tag{24}$$

The corresponding eigenvectors are (in column form, not normalized):

$$U_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1 & 1 & 1 & \dots & 0 \\ 1 & \hat{\omega} & \hat{\omega}^2 & \hat{\omega}^3 & \dots & 0 \\ 1 & \hat{\omega}^2 & \hat{\omega}^4 & \hat{\omega}^6 & \dots & 0 \\ . & . & . & . & . & . \\ 1 & \hat{\omega}^{(q-2)} & \hat{\omega}^{2(q-2)} & \hat{\omega}^{3(q-2)} & \dots & 0 \end{pmatrix} \tag{25}$$

The position of the empty row-column in Eq. (21) depends on the color of the core tag site sandwiched between a and c. Above, the color 0 was assumed. The position of the nill row/column changes according to the actual color. To make this distinction explicit, the corresponding transfer and eigenvector matrix will be denoted by T_L and U_L, respectively. The total number of configurations is given by

$$\Omega = 1680 \times Tr\left\{T_0 T_1 T_2 T_3 T_4 T_5\right\} \tag{26}$$

where we use Ω for the partition function Z at $T = 0$. Ω is the microcanonical partition function and the entropy (alias storage capacity) reads (in bits):

$$S = \log_2 \Omega \tag{27}$$

Since the choice of 6 core outer colors (and thus of the T indices) is arbitrary, so is the order of the matrices in the product. Since Eq. (26) involves sums over all such colors, the result must be invariant under a change of color definitions (or T index permutations). We perform this calculation for 8×8 matrices with the result

$$\Omega(q=8) = 3360\left(23328A^6 + 19440A^5B + 6642A^4B^2 + 1169A^3B^3 + 105A^2B^4 + 6AB^5 + B^6\right) \tag{28}$$

Similar calculations can be performed for other integer values of q, after enlarging the matrix accordingly.

For the Tag 2 case the kernel calculation leads to $A = q + 3v = q - 3$ and $B = v^2 = 1$. Depending on q we obtain after multiplying with the core partition function (1680): $\Omega(q = 8) = 1443288104160$ which corresponds to 40.392496 bits. Note, however, that on a 64 bit

machine (without using arbitrary precision packages), this type of direct calculation can succeed only if $\log_2 \Omega < 64$ bits.

For a q-dependent analytic result we suggest using the spectral approximation method. The spectral decomposition of a matrix T is easiest displayed in the Dirac's bra-ket formalism:

$$T_x = \sum_{n=0}^{q-1} |n_x\rangle \lambda_{n_x} \langle n_x| \tag{29}$$

where $\langle n|$ is the transposed-adjoint of the eigenvector defined in Eq. (25) with $\hat{\omega}^* \leftarrow \hat{\omega}$ and the scalar product is normalized, $\langle n|n\rangle = 1$.

Inserting this form in the Eq. (26)

$$Tr\left\{T_0 T_1 T_2 T_3 T_4 T_5\right\} = \sum_{n_0=0}^{q-1} \sum_{n_1=0}^{q-1} \cdots \sum_{n_5=0}^{q-1} \langle n_0|n_1\rangle\langle n_1|n_2\rangle \ldots \langle n_5|n_0\rangle \lambda_{n_0}\lambda_{n_1} \ldots \lambda_{n_5} \tag{30}$$

Next, we order the eigenvalues in decreasing order and compute this way the leading contributions to the partition function. The largest eigenvalue of any of the T_L corresponds to the single eigenvector with all positive elements, called ground state in physics. It follows that the largest contribution is proportional to λ_0^6. Two ground state eigenvectors of x, y transfer matrices have a scalar product of $\langle 0_x|0_y\rangle = \frac{q-2}{q-1}$, which includes the normalization constant $q-1$. Therefore,

$$\Omega_0 = 1680 \times \left[\frac{\lambda_0(q-2)}{(q-1)}\right]^6 = 1680 \times \left[\frac{(q-2)(q^2-5q+7)}{(q-1)}\right]^6 \tag{31}$$

which corresponds to 39.105 bits for $q = 8$. Note that the difference to the exact result is less than 1 bit.

The first correction to this term is obtained by exchanging one of the leading eigenvalues for a lower one of indices $\{1, 2, \ldots, q-2\}$. When following the corresponding Eq. (30) path, one has instead $\langle 0_x|0_y\rangle\langle 0_y|0_x\rangle = \left(\frac{q-2}{q-1}\right)^2$ the term $\langle 0_x|1_y\rangle\langle 1_y|0_x\rangle$, which equals $\left(\frac{1}{q-1}\right)^2$. Hence, the first correction is

$$\Omega_1 = \frac{6(q-2)\Omega_0}{(q-2)^2(q^2-5q+7)} = \frac{6\Omega_0}{(q-2)(q^2-5q+7)} \tag{32}$$

where 6 accounts for the number of different paths (position of λ_i, $i > 0$) with the same contribution and $q-2$ to the possible index $i > 0$, $i < q-1$ of λ_i. This results shows that for large enough q the spectral approximation is strongly convergent.

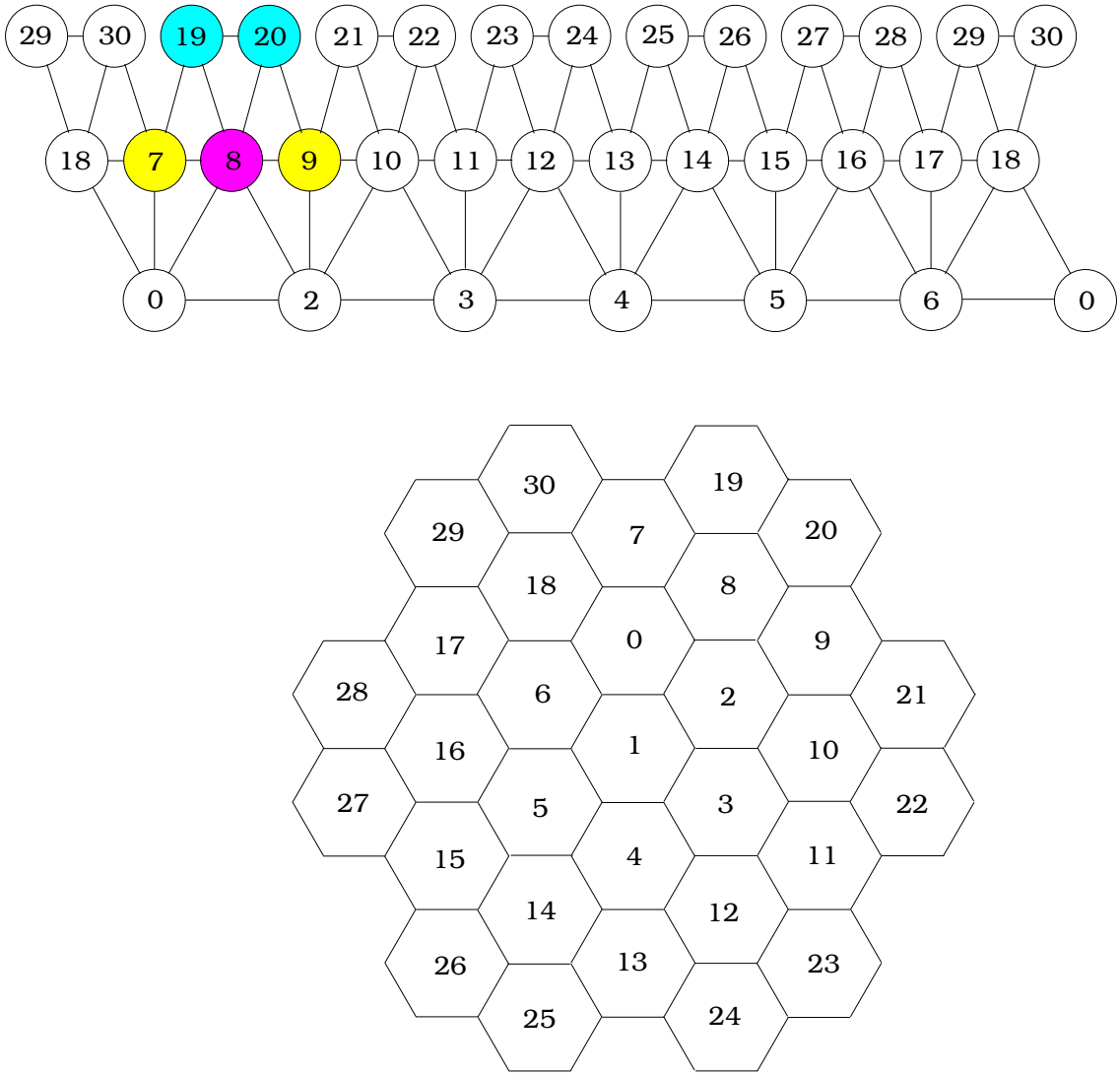

FIG. 5. *The three layers tag (lower part) and its interaction model interpretation (upper part).*

The reference tag 2 has a capacity of 40 bits. Storing a full GPS co-ordinate with 30 cm resolution requires only slightly more. This suggests that the this type of reference tags might be used to expose geodata on places without satellite access or for GPS calibration purposes.

V. VISUAL REFERENCE TAGS WITH 3 LAYERS

Next, consider the visual reference tag with 3 layers shown in Fig. 5. This is the "heavy duty" tag among the visual reference tags. One can think about this tag as "glueing" together 7 core tags. From the Fig. 5 one can see that the cyan colored sites 19 and 20 are

connected only to the two yellow (sites 7,9) and the magenta site 8. Following the previous strategy, we will sum up exactly the sites 19, 20. This process will generate a complex kernel interaction between the second layer sites. However, after some lengthy calculations following the same strategy as in the previous Section, one recovers for the the transfer matrix Eq. (21) the kernel

$$k_3 = (q-3)\left\{\delta_{a,c,x}(q^2-6q+10) + \delta_{a,c}(q-3) + [\delta_{a,x}+\delta_{c,x}](-q^2+5q-7) + q^2 - 5q + 7\right\} \tag{33}$$

where we used $\delta_{a,b} = \delta(a,b)$ in order to fit the equation on a single line.

It is easy to check that $A + C = 0$ and $A + B + 2C + D = 0$. Hence, we are back to the form Eq. (21) but with the constants $A = (q-3)(q^2-5q+7)$ and $B = (q-3)^2$. Hence, $\lambda_0 = (q-1)A + B = (q-1)(q-3)(q^2-5q+7) + (q-3)^2$ and the main contribution to the partition function is

$$\Omega_0 = 1680 \times \left\{\frac{(q-2)(q-3)\left[(q-1)(q^2-5q+7)+q-3\right]}{(q-1)}\right\}^6 \tag{34}$$

For $q = 8$ this first approximation delivers a storage capacity of 70.07795 bits. Note that the exact calculations of the trace can be performed using extended precision packages. Again, the zero order spectral approximation is excellent when compared to the exact result, $\Omega = 1241320861304632500000$ or 70.072366 bits for $q = 8$.

A reference tag sporting 70 bits has enough storage for all possible http(s) address references or database access one might wish.

VI. THE NOISE MODEL AND OPTIMAL DECODING

What kind of errors can occur during the capture and when decoding the image of a visual tag? Since we use black and white among the set of colors, the image processing part will perform a local color calibration based on white balancing or on a Bayesian color match. Errors might occur when the difference between two neighboring colors is not large enough and a misclassification does not contradict the exclusion rule. Mathematically, one can describe the channel noise with the distribution:

$$p(l_i|l_j) = d\delta(l_i,l_j) + m\delta(l_i,l_{j-1}) + m\delta(l_i,l_{j+1}) \tag{35}$$

and $d + 2m = 1$. This equation says that during capture and decoding a given color will be correctly detected with probability d and will be moved one color up or down on the hue angle with probablity m. The actual value of the parameter m can be estimated decoding a set of known reference tags (the *reference* reference tags). In practice, the decoder receives from the image processor a color field, where each color carries also an estimate of their probability m. The optimal decoding strategy is to consider the total Hamiltonian constructed by multiplying the generic interaction Hamiltonian discussed so far with the site-contributions Eq. (35).

Based on this Hamiltonian, the decoding strategy generating in average the minimal number of errors is to evaluate the marginal color distribution for each site and then choose the most probable value. This calculation can be performed either analytically by extending accordingly the calculations of the previous Sections or by performing a Monte Carlo simulation, or a combination of both. The interested Reader can consult the PhD Thesis of my graduate student Thorsten Wanschura [4], who performed an optimal Bayesian reconstruction of images using this technique, with spectacular results.

While the design of the reference tags has taken into account the requirements posed by the capture and image processing part before the decoding process, there might be situations where a stronger error correction (*ecc*) mechanism is needed. For such cases one should use the *ecc* encoder on the data stream *before* reaching the tag encoder and the *ecc* decoder *after* the visual tag has been reconstructed. That decouples the error correcting mechanism from the image capture and tag decoding process. Of course, the net information content stored in the tag will be reduced but so would be it when implementing such a ecc code directly in the tag.

VII. CONCLUSIONS

In this paper we calculated analytically the storage capacity of several visual reference tags, which are small circular clusters of hexagons obeying nearest neighbor graph coloring rules. Since the work of Temperley and Lieb [2] it is known that some antiferromagnetic Potts models can be mapped into ice models, and some of those can be exactly solved [3]. This explains in part why an analytic calculation is feasible at all for relatively large clusters and an arbitrary number of $Z(q)$ states. For the other part, I am indebted to Anthony C.

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